GLOBAL REPRESENTATIONS OF THE HEAT AND SCHRÖDINGER EQUATION WITH SINGULAR POTENTIAL

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Abstract. We study the n-dimensional Schrödinger equation with singular potential $V_\lambda(x) = \lambda \|x\|^{-2}$. Its solution space is studied as a global representation of $\widetilde{SL(2,\mathbb{R})} \times O(n)$. A special subspace of solutions for which the action globalizes is constructed via nonstandard induction outside the semisimple category. The space of K-finite vectors is calculated, obtaining conditions for λ so that this space is non-empty. The direct sum of solution spaces, over such admissible values of λ is studied as a representation of the $2n+1$-dimensional Heisenberg group.

1. Introduction

The Schrödinger and heat equations have been heavily studied in physics and in mathematics. In [4], the solutions to a family of differential equations, including the potential free Schrödinger and heat equations, were examined as global representations of the corresponding Lie symmetry group, $G := (\widetilde{SL(2,\mathbb{R})} \times O(n)) \ltimes H_{2n+1}$, where $\widetilde{SL(2,\mathbb{R})}$ denotes the two-fold cover of $SL(2,\mathbb{R})$ and H_{2n+1} denotes the $2n+1$-dimensional Heisenberg group. Using the same techinques, an invariant subspace of solutions to the one-dimensional Schrödinger equation with singular potential $V(x) = \lambda/x^2$ was studied in [3]. The natural generalization of these two articles is the goal of this article.

Let $x \in \mathbb{R}^n$, $s \in \mathbb{C}$, and Δ_n denote the Laplacian operator on $\mathbb{R}^n$. Here we study the representation theory associated to special subspaces of solutions to the family of differential equations of the form

$$4s\partial_t + \Delta_n = \frac{2\lambda}{\|x\|^2}, \tag{1.1}$$

that are invariant under the action of the group G. By letting $s = -i/2$ or $s = -1/4$ one obtains the Schrödinger or the heat equation with singular potential, respectively. In terms of representation theory, this problem will be equivalent to the solution of an eigenvalue problem to a Casimir element in a certain line bundle over a compactification of $\mathbb{R}^{1,n}$.

In more detail, we start by constructing an induced representation space that carries the structure of a global Lie group representation. To do that, we consider a parabolic-like subgroup $\overline{P}$ of G and the smoothly induced representation

$$I(q,r,s) = \mathrm{Ind}_{\overline{P}}^{G}(\chi_{q,r,s})$$

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where $\chi_{q,r,s}$ is a character on $\overline{P}$ with parameters $r, s \in \mathbb{C}$ and $q \in \mathbb{Z}_4$ (See Section 2.2).

Let $\Omega' = 2\Omega\mathfrak{sl}(2,\mathbb{R}) - \Omega_{\mathfrak{so}(n)} - r(r+2)$ where $\Omega_{\mathfrak{sl}(2,\mathbb{R})}$ is the Casimir element of $\mathfrak{sl}(2,\mathbb{R})$ and $\Omega_{\mathfrak{so}(n)}$ is the Casimir element for $\mathfrak{so}(n)$ in the universal enveloping algebra of Lie(G). Using the fact that $\mathbb{R}^{1,n}$ can be embedded as an open dense set in $G/\overline{P}$, we realize smooth sections in $I(q,r,s)$ and use this realization to show that the kernel of $\Omega' - 2\lambda$ in this space is a solution subspace of (1.1).

In order to study the structure of $\ker(\Omega' - 2\lambda)$, we turn to the analog of what would be the compact picture in the semisimple category. There we explicitly derive the conditions, on λ, for the existence of K-finite vectors (see Theorem 1). Moreover, the form of the K-finite vectors is given explicitly in terms of confluent hypergeometric functions of the first kind and harmonic polynomials.

We will say that the eigenvalue λ is admissible if and only if the space of K-finite vectors, $\ker(\Omega' - 2\lambda)_K$, is non-trivial. The set of all non-zero admissible eigenvalues will be denoted by A_n and is explicitly determined in Section 7. For each $\lambda \in A_n$, $\ker(\Omega' - 2\lambda)_K$ decomposes as a direct sum of finitely many infinite dimensional representations, whose structure as $\mathfrak{sl}_2 \times O(n)$-modules is completely determined, including structures of highest or lowest weight modules.

When $\lambda = 0$, $\ker \Omega'$ is invariant under the action of the Heisenberg algebra and its structure is determined in [4]. However, for non-zero values of λ, the action of the Heisenberg algebra does not preserve $\ker(\Omega' - 2\lambda)_K$. Nevertheless, we show that the space

$$\ker \Omega' \oplus \bigoplus_{\lambda \in A_n} \ker(\Omega' - 2\lambda)_K$$

carries the structure of a $\mathfrak{g}$-module, where $\mathfrak{g}$ denotes the Lie algebra of G. The composition series of this space is determined in Theorem 3.

2. Notation

We will follow the constructions in [4] very closely.

2.1. **The Group.** For $x, y \in \mathbb{R}^{2n}$, let $\langle x, y\rangle = x^T J_n y$ where $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Let H_{2n+1} denote the $(2n+1)$-dimensional Heisenberg group. The multiplication in H_{2n+1} is given by

$$(v,t)(v',t') = (v+v', t+t' + \langle v, v'\rangle)$$

where $v, v' \in \mathbb{R}^{2n}$ and $t, t' \in \mathbb{R}$.

An element $\sigma \in Sp(2n,\mathbb{R})$ acts on H_{2n+1} by $\sigma.(v,t) = (\sigma.v, t)$ where the action $\sigma.v$ is the standard action of $Sp(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$. Thus we can define the product on $Sp(2n,\mathbb{R}) \ltimes H_{2n+1}$ as

$$(\sigma, h)(\tau, k) = (\sigma\tau, \tau^{-1}(h)k)$$

for $\sigma, \tau \in Sp(2n,\mathbb{R})$ and $h, k \in H_{2n+1}$.

The group $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ can be embedded in $Sp(2n,\mathbb{R})$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} aI_n & bI_n \\ cI_n & dI_n \end{smallmatrix}\right)$ and the group $O(n)$ can be embedded diagonally by $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right)$. Since these two images commute, there exists a homomorphism $B : SL(2,\mathbb{R}) \times O(n) \to Sp(2n,\mathbb{R})$ with kernel $\pm(I_2 \times I_n)$.

Following the realization of the two-fold cover of $SL(2,\mathbb{R})$ in [4], define the complex upper half plane $D := \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ and let $SL(2,\mathbb{R})$ act on D by

fractional linear transformations, that is, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ and $z \in D$ then

$$g.z = \frac{az+b}{cz+d}.$$

Define $d : SL(2,\mathbb{R}) \times D \to \mathbb{C}$ by $d(g,z) := cz + d$. Then there are exactly two smooth square roots of $d(g,z)$ for each $g \in SL(2,\mathbb{R})$ and $z \in D$. The double cover can be realized as:

$$\begin{aligned} G_2 := \{(g,\epsilon) | g \in SL(2,\mathbb{R}) \text{ and smooth } \epsilon : D \to \mathbb{C} \\ \text{such that } \epsilon(z)^2 = d(g,z) \text{ for } z \in D\} \end{aligned}$$

with the product defined by

$$(g_1, \epsilon_1(z))(g_2, \epsilon_2(z)) = (g_1 g_2, \epsilon_1(g_2.z)\epsilon_2(z)).$$

Let $p : G_2 \to SL(2,\mathbb{R})$ be the canonical projection. Then $B \circ (p \otimes 1) : G_2 \times O(n) \to Sp(2n,\mathbb{R})$ is a homomorphism and the semidirect product

$$G := (G_2 \times O(n)) \ltimes H_{2n+1}$$

is well-defined via this homomorphism.

2.2. **Parabolic Subgroup and Induced Representations.** Let $\exp_{G_2} : \mathfrak{sl}(2,\mathbb{R}) \to G_2$ denote the exponential map. Let $\mathfrak{a} = \{\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right) | t \in \mathbb{R}\}$, $\mathfrak{n} = \{\left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right) | t \in \mathbb{R}\}$, and $\overline{\mathfrak{n}} = \{\left(\begin{smallmatrix} 0 & 0 \\ t & 0 \end{smallmatrix}\right) | t \in \mathbb{R}\}$. Then,

$$\begin{aligned} A := \exp_{G_2}(\mathfrak{a}) &= \{(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right), z \mapsto e^{-t/2}) | t \in \mathbb{R}^{\geq 0}\} \\ N := \exp_{G_2}(\mathfrak{n}) &= \{(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), z \mapsto 1) | t \in \mathbb{R}\} \\ \overline{N} := \exp_{G_2}(\overline{\mathfrak{n}}) &= \{(\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right), z \mapsto \sqrt{tz+1}) | t \in \mathbb{R}\}. \end{aligned}$$

Let $\mathfrak{k} := \{\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right) : \theta \in \mathbb{R}\}$ then

$$K_2 := \exp_{G_2}(\mathfrak{k}) = \{(g_\theta, \epsilon_\theta) := (\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right), z \mapsto \sqrt{\cos\theta - z\sin\theta}) | \theta \in \mathbb{R}\}$$

where $\sqrt{\cdot}$ denotes the principal square root in $\mathbb{C}$. Writing M for the centralizer of A in K then

$$M = \{m_j := (\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^j, z \to i^{-j}) | j = 0,1,2,3\}.$$

Let $W \subset H_3$ be given by $W = \{(0,v,w) | v \in \mathbb{R}^n \text{ and } w \in \mathbb{R}\} \cong \mathbb{R}^{n+1}$ and let $\overline{P} = (MA\overline{N} \times O(n)) \ltimes W$.

It is well known that the character group on A is isomorphic to the additive group $\mathbb{C}$ so any character on A can be indexed by a constant $r \in \mathbb{C}$ and defined by

$$\chi_r\big((\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right), z \mapsto e^{-t/2})\big) = t^r$$

for $t > 0$. A character on M is parametrized by $q \in \mathbb{Z}_4$ and defined by $\chi_q(m_j) = i^{jq}$. A character on W can be parametrized by $s \in \mathbb{C}$ and defined by,

$$\chi_s\big((0,v,w)\big) = e^{sw}.$$

Finally, any character on $\overline{P}$ that is trivial on N is parametrized by a triplet (q,r,s) where $s, r \in \mathbb{C}$ and $q \in \mathbb{Z}_4$ and defined by

$$\chi_{q,r,s}\Big(((-1)^j \left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right), z \mapsto i^{-j} e^{-a/2}\sqrt{acz+1}), (0,v,w)\Big) = i^{jq}|a|^r e^{sw}. \tag{2.1}$$

The representation space induced by $\chi_{q,r,s}$ will be denoted by $I(q,r,s)$ and defined by

$$I(q,r,s) := \{\phi : G \to \mathbb{C} | \phi \in C^\infty \text{ and } \phi(g\overline{p}) = \chi_{q,r,s}^{-1}(\overline{p})\phi(g) \text{ for } g \in G, \overline{p} \in \overline{P}\}$$

the action of G on $I(q,r,s)$ is given by left translation: $(g_1.\phi)(g_2) = \phi(g_1^{-1}g_2)$.

3. The Non-compact Picture

If $X := \{(x,0,0)|x \in \mathbb{R}^n\}$, then $H_3 = XW$ and $G = (N \times X)\overline{P}$ a.e. Since $N \times X$ is isomorphic to $\mathbb{R}^{n+1}$ via $(t,x) \mapsto N_{t,x} := [((\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}), z \mapsto 1), (x,0,0)]$ and a section in the induced representation, $I(q,r,s)$, is determined by its restriction to N, there exists an injection of $I(q,r,s)$ into $C^\infty(\mathbb{R}^{n+1})$, given by restriction of domain. The image of this map is identified as

$$I'(q,r,s) = \{f \in C^\infty(\mathbb{R}^{n+1}) | f(t,x) = \phi(N_{t,x}) \text{ for some } \phi \in I(q,r,s)\}$$

and given the G-module that makes the map $\phi \mapsto f$ intertwining.

The actions of G and the correspondent action of $\mathfrak{g}$ on $I'(q,r,s)$ have been calculated in [4]. We will record these results, since they are used in later sections.

Proposition 1. *Let $f \in I'(q,r,s)$, $(g,\epsilon) \in G_2$, and $(v_1,v_2,w) \in H_{2n+1}$. Then,*

$$((g,\epsilon).f)(t,x) = (a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z))e^{\frac{-sc\|x\|^2}{a-ct}} f\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right) \tag{3.1a}$$

(3.1b)

$$((v_1,v_2,w).f)(t,x) = e^{-s(v_1\cdot v_2 - 2v_2\cdot x - t\|v_2\|^2 + w)} f(t, x - v_1 + tv_2).$$

Let $u \in O(n)$ then $(u.f)(t,x) = f(t, u^{-1}x)$.

Corollary 1. *The action of $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$ on $I'(q,r,s)$ is given by the differential operator*

$$(ct-a)\sum_{j=1}^n x_j\partial_j + (ct^2 - 2at - b)\partial_t + (ra - cs\|x\|^2 - rct). \tag{3.2}$$

An element $(u,v,w) \in \mathfrak{h}_{2n+1}$ acts on $I'(q,r,s)$ by the differential operator

$$-\sum_{j=1}^n u_j\partial_j + t\sum_{j=1}^n v_j\partial_j + s(w - 2v\cdot x).$$

Proof. It follows from differentiating the group actions on $I'(q,r,s)$. □

4. Casimir Operators

Let

$$E_n = \sum_{j=1}^n x_j\partial_j$$

denote the Euler operator on $\mathbb{R}^n$,

$$\Omega_{\mathfrak{sl}(2,\mathbb{R})} = \frac{1}{2}h^2 - h + 2e^+e^-$$

denote the Casimir element in the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$, and $\Omega_{\mathfrak{so}(n)}$ denote the Casimir element for $\mathfrak{so}(n)$. In [4] it was shown that the element Ω in the universal enveloping algebra of $\mathfrak{g}$ defined by

$$\Omega = 2\Omega_{\mathfrak{sl}(2,\mathbb{R})} - \Omega_{\mathfrak{so}(n)} - r(r+2)$$

acts on $I'(q,r,s)$ as the differential operator

$$\Omega = -(2r+n)E_n + \|x\|^2 (4s\partial_t + \Delta_n)$$

where Δ_n denotes the n-th dimensional Laplacian.

In particular, for $r=-n/2$, Ω acts by $\Omega = \|x\|^2 (4s\partial_t + \Delta_n)$ so that,

$$\ker(\Omega - 2\lambda) = \ker\left(4s\partial_t + \Delta_n - \frac{2\lambda}{\|x\|^2}\right).$$

Remark 1. *In case $r=-n/2$ and $s=-1/2$ (respectively, $s=-i/2$) the invariant subspace $\ker(\Omega-2\lambda)$ is contained in the space of solutions of the heat (respectively, Schrödinger equation) with singular potential $\frac{2\lambda}{\|x\|^2}$. From here on, we let $r=-n/2$.*

5. The Compact Picture

We have exposed the intertwining isomorphism between $I(q,-\frac{n}{2},s)$ and the non-compact picture $I'(q,-\frac{n}{2},s)$. In this section, we realize $I(q,-\frac{n}{2},s)$ in a way that will allow us determine the $K_2\times O(n)$-weight vectors explicitly. To that end, we recall that the group G_2 has Iwasawa decomposition $G_2 = K_2 A\overline{N}$ and notice that the product induces a diffeomorphism $G \cong (K_2\times X)\times((A\overline{N}\times O(n))\ltimes W)$. Since $((A\overline{N}\times O(n))\ltimes W)\subset \overline{P}$, an element $\phi\in I(q,-\frac{n}{2},s)$ is completely determined by its restriction to $K_2\times X$.

Since $K_2\cong S^1$ via a 4π-periodic isomorphism, smooth functions on $K_2\times X$ can be realized as smooth functions on $S^1\times\mathbb{R}^n$, which can be extended by continuity and periodicity to smooth functions on $\mathbb{R}^{n+1}$. Therefore, the image of the restriction can be realized as a map from $I(q,-\frac{n}{2},s)$ to $C^\infty(\mathbb{R}^{n+1})$ and given by $\varphi\mapsto F$ iff

$$\phi([(g_\theta,\epsilon_\theta),(y,0,0)]) = F(\theta,y).$$

We denote the image of this map $I''(q,-\frac{n}{2},s)$. This space is provided with the structure of a G-module that makes the map intertwining. Calculating the action of $(g_\theta,\epsilon_\theta)$ and considering the periodicity conditions required, it can be shown that

$$I''(q,-\frac{n}{2},s) = \{F\in C^\infty(\mathbb{R}^{n+1}) | F(\theta+j\pi,(-1)^j y) = i^{-jq}F(\theta,y)\}. \tag{5.1}$$

Since we established an isomorphism between $I(q,-\frac{n}{2},s)$ and $I'(q,-\frac{n}{2},s)$, there exists an induced isomorphism from $I'(q,-\frac{n}{2},s)$ to $I''(q,-\frac{n}{2},s)$. This isomorphism is given by $f\mapsto F$ iff

$$f(t,x) = (1+t^2)^{-n/4} e^{\frac{st\|x\|^2}{1+t^2}} F(\arctan t, x(1+t^2)^{-1/2}). \tag{5.2}$$

equivalently, one can define $F\mapsto f$ iff

$$F(\theta,y) = (\cos\theta)^{-n/2} e^{-s\|y\|^2\tan\theta} f(\tan\theta, y\sec\theta) \tag{5.3}$$

for $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$ noting that F can be extended to $\theta\in\mathbb{R}$ extending to the end points by continuity using $F(\theta+j\pi,(-1)^j y) = i^{-jq}F(\theta,y)$.

Under this isomorphism, via the chain rule, we obtain

$$\partial_t \leftrightarrow \frac{1}{2}(-y\sin 2\theta\partial_y + \cos^2\theta\partial_\theta + 2sy^2\cos 2\theta - 1/2r\sin 2\theta) \tag{5.4a}$$

$$\partial_x \leftrightarrow 2sy\sin\theta + \cos\theta\partial_y. \tag{5.4b}$$

Define a standard basis of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$\kappa = i(e^- - e^+)$$

and

$$\eta^{\pm} = 1/2(h \pm i(e^+ + e^-)).$$

Applying equations (5.4) to the action in Corollary 1, it can be shown that the $\mathfrak{sl}_2$-triple just defined acts on $I''(q, -\frac{n}{2}, s)$ by the differential operators

$$\kappa = i\partial_\theta \tag{5.5}$$

$$\eta^{\pm} = \frac{1}{2} e^{\mp 2i\theta} \left(-E_n \mp i\partial_\theta - (n/2 \pm 2is \left\| y \right\|^2) \right). \tag{5.6}$$

In the following corollary, we use these actions to calculate the action of Ω' on $I''(q, -\frac{n}{2}, s)$.

Proposition 2. *If Ω'' denotes the differential operator by which the central element Ω' acts on $I''(q, -\frac{n}{2}, s)$ then*

$$\Omega'' = \left\| y \right\|^2 \left(4s\partial_\theta + 4s^2 \left\| y \right\|^2 + \Delta_n \right).$$

6. $K_2 \times O(n)$-types in $\ker(\Omega'' - 2\lambda)$

The goal of this section is to write the K-types in $\ker(\Omega''-2\lambda)$ explicitly. Let $K = K_2 \times O(n)$ and write $\mathcal{H}_k(\mathbb{R}^n)$ for the space of harmonic polynomials of homogeneous degree k and $\mathcal{H}_k(S^{n-1})$ for the restriction of elements of $\mathcal{H}_k(\mathbb{R}^n)$ to S^{n-1}. The $O(n)$-finite vectors in $C^\infty(S^{n-1})$ are the harmonic polynomials on S^{n-1}. That is $C^\infty(S^{n-1})_{O(n)-finite} = \bigoplus_k \mathcal{H}_k(S^{n-1})$, where $k \in \mathbb{Z}$ for $n = 2$, $k \in \mathbb{Z}^{\geq 0}$ for $n \geq 3$, and $k \in \{0, 1\}$ for $n = 1$. We decompose $0 \neq y \in \mathbb{R}^n$ in polar coordinates as $y = \rho\xi$ with $\rho = \|y\|$ and $\xi \in S^{n-1}$.

Proposition 3. *The space of K-finite vectors in $I''(q, -\frac{n}{2}, s)$ is the span of all functions of the form*

$$F(\theta, y) = e^{-im\theta/2} \psi(\rho) h_k(y),$$

where $m \in \mathbb{Z}$, $\psi \in C^\infty(0, \infty)$, $h_k \in \mathcal{H}_k(\mathbb{R}^n)$, and

$$m \equiv q + 2k \mod 4$$

for $y \neq 0$ so that $F(\theta, y)$ extends smoothly to $y = 0$ and $\lim_{\rho \to 0} \rho^k \psi(\rho)$ is bounded.

Proof. This result is proved in [4]. □

Lemma 1. *If $m \in \mathbb{Z}$, $\psi \in C^2(\mathbb{R})$, and $h \in H_k(\mathbb{R}^n)$, then a function of the form $F(\theta, y) = e^{-im\theta/2}\psi(\rho)h_k(y)$ is in $\ker(\Omega'' - 2\lambda)$ if and only if ψ is annihilated by the differential operator*

$$\mathcal{D} = \rho^2 \partial_\rho^2 + (n - 1 + 2k)\rho\partial_\rho + 4s^2\rho^4 - 2ism\rho^2 - 2\lambda$$

Proof. The result follows from the following calculation:

$$\rho^2 \Delta_n(\psi h) = \rho^2 \psi'' h + \rho((n-1)h + 2kh)\psi'$$

together with the fact that

$$\Omega'' - 2\lambda = \rho^2 \left(4s\partial_\theta + 4s^2\rho^2 + \Delta_n - \frac{2\lambda}{\rho^2} \right).$$

□

In preparation for writing the K-finite vectors explicitly and determining the conditions for their existence, we state the following lemma.

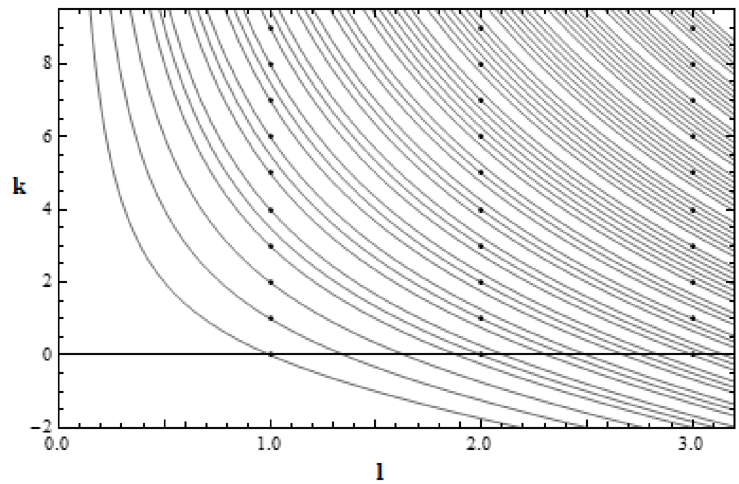

FIGURE 1. Admissible λ-level curves for $n = 3$ in the (l, k)-plane.

Lemma 2. *Let*

$$A_n := \{l(n+2j) \mid l, j \in \mathbb{Z} \text{ and } 1 \leq l \leq j+1\}$$

If $n \geq 2$ is even, then $A_n = 2\mathbb{Z}^{\geq n}$. If $n \geq 3$ is odd, then

$$A_n = \{\gamma = 2^r a \in \mathbb{Z}^{\geq 0} \mid (a, 2) = 1 \text{ and } a \geq n + 2^{r+1} - 2\}$$

Proof. The case when n is even is clear by letting $l = 1$. So, let $n \geq 3$ be odd and write $l = 2^r b$ where $(b, 2) = 1$ and $b \geq 1$ then $l(n+2j) = 2^r b(n+2j) = 2^r a$ where $a = b(n+2j)$. Now, $b(n+2j) \geq b(n+2(l-1)) = b(n+2(2^r b-1)) \geq n+2^r+1-2$. So right hand side is contained in A.

The other inclusion follows almost immediately, by noticing that $\gamma = 2^r(1+2k) = 2^r(n+(2k+1-n)) \geq 2^r(n+2^{r+1}-2)$. Therefore we have $1 \leq 2^r \leq \frac{2k-n+1}{2}+1$, thus $\gamma \in A$. $\square$

For convenience, we state the following definition. It will be used to describe the eigenvalues for which the space of K-finite vectors is non-empty. We must remark that it should not be confused with the standard usage of the term admissible for a representation.

Definition 1. (1) *For $n = 1$ we say that an **eigenvalue** λ **is admissible** if and only if λ is a triangular number. That is $\lambda = \frac{1}{2}l(l-1)$ for some $l \in \mathbb{Z}^{\geq 0}$. Denote this set by A_1.*
(2) *For $n \geq 2$ we say that an **eigenvalue** λ **is admissible** if and only if $\lambda \in A_n$.*
(3) *Let $n = 2$ and λ be admissible. The pair $(l, k) \in \mathbb{Z} \times \mathbb{Z}^{\geq 1}$ is λ-**admissible** if and only if $\lambda = l(2l+2k+n-2)$.*
(4) *Let $n \geq 3$ and $\lambda \in A_n$. We call the pair $(l, k) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 1}$, λ-**admissible** if and only if $\lambda = l(2l+2k+n-2)$.*
(5) *When $n = 1$, a pair $(l, 0)$ is λ-admissible iff $\lambda = \frac{l(l-1)}{2}$.*

In Figure 1, the admissible λ level curves for $n=3$ are represented. Notice that for $\lambda=0$, the corresponding level curve is the y-axis. The λ-admissible pairs (l,k) are the points in $\mathbb{Z}^{\geq 1}\times\mathbb{Z}^{\geq 0}$ that intersect the λ level curve.

Theorem 1. (1) *The space of K-finite vectors in* $\ker(\Omega''-2\lambda)\subset I''(q,-\frac{n}{2},s)$ *is non-empty if and only if λ is admissible.*

(2) *In this case, the space of K-finite vectors in* $\ker(\Omega''-2\lambda)\subset I''(q,-\frac{n}{2},s)$ *is spanned by functions of the form*

$$F_{m,l,k}(\theta,y)=e^{-im\theta/2}e^{-is\rho^2}\rho^{2l}h_k(y)\,{}_1F_1\Big(\frac{m+4l+2k+n}{4},2l+k+n/2,2is\rho^2\Big),$$

where the pair (l,k) is λ-admissible and $m\equiv 2k+q \mod 4$.

Proof. The one dimensional case was analyzed in [3] and the case when $\lambda=0$ has been studied in [4]. So, let $n\geq 2$ and $\lambda\neq 0$. The elements in $\ker(\Omega''-2\lambda)\subset I''(q,-\frac{n}{2},s)$ are of the form $F(\theta,y)=e^{-im\theta/2}\psi(\rho)h_k(y)$, they satisfy the conditions in Proposition 3, and by Lemma 1, $\mathcal{D}\psi=0$. Following [2], the Frobenius method for this equation yields a solution spanned by two linearly independent solutions. The indicial roots for this equation are

$$r_\pm=\frac{1}{2}[2-2k-n\pm\sqrt{(n-2+2k)^2+8\lambda}].$$

Since $\mathcal{D}$ respects the decomposition of ψ in terms of its even and odd components and these are determined by their value on $\mathbb{R}^{\geq 0}$ we may assume that ψ is a function of ρ^2. Then, the first linearly independent solution is of the form

$$\psi(\rho)=\rho^{r_+}(1+\sum_{j=1}^{\infty}c_j(r_+)\rho^{2j})$$

for some $c_j(r_+)\in\mathbb{R}$. This function extends to a smooth function of $y\in\mathbb{R}^n$ if and only if $r_+\in 2\mathbb{Z}^{\geq 0}$, equivalently if and only if

$$\lambda=l(2l+2k+n-2)$$

for $l\in\mathbb{Z}^{\geq 0}$ where $r_+=2l$. The parity conditions of λ stated in (1) follow directly from this expression together with Lemma 2. Moreover, it is clear that l must be a divisor of λ. Solving for k one obtains $k=\frac{\lambda}{2l}-l+1-\frac{n}{2}$. Then $k\in\mathbb{Z}$ iff $\lambda/l\equiv n \mod 2$ and when $n\geq 3$ we need $k\geq 0$, this is so if and only if $1\leq l\leq 1/4(-(n-2)+\sqrt{(n-2)^2+8\lambda})$.

If $r_+\in 2\mathbb{Z}^{\geq 0}$, then $r_+-r_-\in Z^{\geq 0}$. If $r_+-r_-\neq 0$ then the second independent solution is of the form $\psi_2(\rho)=a\psi(\rho)\ln\rho+\rho^{r_-}(1+\sum_{j=0}^{\infty}c_j(r_-)\rho^{2j})$ and it is not continuous at zero because $r_-<0$. If $r_+-r_-=0$ then $\lambda=0$ and $(k,n)=(0,2)$ is the only possible solution. In this case, it is known that $a=1$ so $\ln\rho$ makes the second solution not continuous at zero. Therefore, for each of these admissible pairs (l,k) there exists a unique K-finite vector of the form $\psi(\rho)=\rho^{2l}(1+\sum_{j=1}^{\infty}c_j\rho^{2j})$.

To establish (2) now, it suffices to show that for fixed (m,l,k), the corresponding $\psi_{m,l,k}(\rho)=e^{-is\rho^2}\rho^{2l}\ {}_1F_1(\frac{m+4l+2k+n}{4},2l+k+n/2,2is\rho^2)$. Explicitly calculating $\mathcal{D}(\rho^{2l}e^{-is\rho^2}F(2is\rho^2))$ one obtains the following differential equation:

$$2is\rho^2F''(2is\rho^2)+(k+2l+\frac{n}{2}-2is\rho^2)F'(2is\rho^2)-\frac{m+4l+2k+n}{4}F(2is\rho^2)=0.$$

Recall that the confluent hypergeometric differential equation is

$$(z\partial_z^2+(b-z)\partial_z-a)\,{}_1F_1(a,b,z)=0$$

(c.f. [1]). This equation has well known solutions in the form of confluent hypergeometric functions of the first and second kind. However, the smoothness condition required by being in $I''(q,r,s)$ shows that the unique solution corresponds to a multiple of the confluent hypergeometric function of the first kind. We may therefore take $F(2is\rho^2) = {}_1F_1(\frac{m+4l+2k+n}{4}, 2l+k+n/2, 2is\rho^2)$. As a function of y, this solution extends smoothly to a solution on $\mathbb{R}^n$. □

7. Irreducible Subspaces of $\ker(\Omega'' - 2\lambda)$

In this section, we look at the structure of $\ker(\Omega''-2\lambda) \subset I''(q,-\frac{n}{2},s)$ as an $\mathfrak{sl}_2 \times O(n)$-module. To that end, we will explicitly compute the actions of the standard $\mathfrak{sl}_2$-basis. For these calculations, we will use the following properties of the confluent hypergeometric function (c.f. [1])

$$\frac{d^n}{dz^n}\,{}_1F_1(a,b,z) = \frac{(a)_n}{(b)_n}\,{}_1F_1(a+n,b+n,z) \tag{7.1a}$$

$$b\,{}_1F_1(a,b,z) - b\,{}_1F_1(a-1,b,z) - z\,{}_1F_1(a,b+1,z) = 0 \tag{7.1b}$$

$$\begin{aligned} b(1-b+z)\,{}_1F_1(a,b,z) + b(b-1)\,{}_1F_1(a-1,b-1,z)& \\ - az\,{}_1F_1(a+1,b+1,z) &= 0 \end{aligned} \tag{7.1c}$$

$$\begin{aligned} (a-1+z)\,{}_1F_1(a,b,z) + (b-a)\,{}_1F_1(a-1,b,z)& \\ (1-b)\,{}_1F_1(a,b-1,z) &= 0 \end{aligned} \tag{7.1d}$$

$$(a-b+1)\,{}_1F_1(a,b,z) - a\,{}_1F_1(a+1,b,z) + (b-1)\,{}_1F_1(a,b-1,z) = 0 \tag{7.1e}$$

Combining (7.1b) with $a+1$ instead of a and (7.1e) one obtains

$$ {}_1F_1(a,b,z) = {}_1F_1(a,b-1,z) - \frac{az}{b(b-1)}\,{}_1F_1(a+1,b+1,z). \tag{7.2}$$

Using Equation (7.1e) with $b+1$ in place of b and combining it with (7.1c), one obtains

$$ {}_1F_1(a,b,z) = {}_1F_1(a-1,b-1,z) - \frac{b-a}{b-1} z\,{}_1F_1(a,b+1,z). \tag{7.3}$$

Theorem 2. *For a λ-admissible pair (l,k) such that $h_k \in \mathcal{H}_k(\mathbb{R}^n)$ is non-zero, we have*

$$\kappa.F_{m,l,k}(\theta,y) = \frac{m}{2}F_{m,l,k} \tag{7.4a}$$

$$\eta^{\pm}.F_{m,l,k}(\theta,y) = -\frac{\pm m+4l+2k+n}{4}F_{m\pm 4,l,k} \tag{7.4b}$$

Lowest weight vectors occur iff $q \equiv n \mod 4$ in this case

$$F_{(2k+4l+n),l,k} = e^{-\frac{1}{2}(2k+4l+n)i\theta}e^{is\rho^2}\rho^{2l}h_k$$

is a lowest weight vector. Highest weight vectors occur iff $q+n \equiv 0 \mod 4$ and in this case

$$F_{-(2k+4l+n),l,k} = e^{\frac{1}{2}(2k+4l+n)i\theta}e^{-is\rho^2}\rho^{2l}h_k$$

is a highest weight vector.

Proof. Let $p_m(\theta,\rho) = e^{-im\theta/2}e^{-is\rho^2}$. Since $\eta^{\pm}$ act by $-1/2e^{\mp 2i\theta}[-E_n \mp i\partial_\theta + (n/2 \mp 2is\rho^2)]$, we have

$$\eta^{+}.F_{m,l,k}(\theta,y) = -p_{m+4}(\theta,\rho)\rho^{2l}h_k(y)\left(a\,{}_1F_1(a,b,z) + z\frac{a}{b}\,{}_1F_1(a+1,b+1,z)\right)$$

where $a = \frac{m+4l+2k+n}{4}$, $b = 2l + k + n/2$, and $z = 2is\rho^2$. Then, Equation (7.1b) implies

$$\eta^+.F_{m,l,k}(\theta, y) = -\frac{m+4l+2k+n}{4} F_{m+4,l,k}.$$

The action of η^- is as follows:

$$\begin{aligned}\eta^-.F_{m,l,k}(\theta, y) &= -1/2 p_{m-4}(\theta,\rho)\rho^{2l} h_k(y)\Big((2l+k-m/2+n/2-4is\rho^2)\,{}_1F_1(a,b,z)\\ &\qquad +2z\frac{a}{b}\;{}_1F_1(a+1,b+1,z)\Big)\\ &= -p_{m-4}(\theta,\rho)\rho^{2l}h_k(y)\,((b-a-z)\,{}_1F_1(a,b,z)\\ &\qquad\qquad +z\frac{a}{b}\;{}_1F_1(a+1,b+1,z)\Big).\end{aligned}$$

Equation (7.1c) implies

$$\eta^-.F_{m,l,k}(\theta, y) = -p_{m-4}(\theta,\rho)\rho^{2l}h_k(y)\,((1-a)\,{}_1F_1(a,b,z) + (b-1)\,{}_1F_1(a-1,b-1,z))$$

Equation (7.1e) implies

$$\eta^-.F_{m,l,k}(\theta, y) = -\frac{-m+4l+2k+n}{4} F_{m-4,l,k}.$$

The statement about the highest and lowest weights follows from observing that these can occur only when $2k+n+4l \equiv m \mod 4$ or $2k+n+4l \equiv -m \mod 4$. This fact, together with the condition that $2k+q \equiv m \mod 4$, gives the desired result. The form of the highest weight vectors follows from directly calculating the weight vectors with weight $m = -(2k+n+4l)$. The form of the lowest weight vectors follows in the same way but with $m = 2k+n+4l$. $\square$

Definition 2. *Let* $\ker(\Omega'' - 2\lambda)_K$ *denote the* K*-finite vectors of* $\ker(\Omega'' - 2\lambda) \subset I''(q, -\frac{n}{2}, s)$. *For a* λ*-admissible pair* (l,k) *define* $H_{l,k} \subset \ker(\Omega'' - 2\lambda)_K$ *by*

$$H_{l,k} = \operatorname{span}\{F_{m,l,k} \mid m \equiv 2k+q \mod 4\}$$

If $q \equiv n \mod 4$, *define* $H^+_{l,k} \subset H_{l,k}$ *by*

$$H^+_{l,k} := \{F_{m,l,k} \mid m \geq (2k+4l+n),\ m \equiv 2k+q \mod 4\ \}.$$

If $q \equiv -n \mod 4$, *define* $H^+_{l,k} \subset H_{l,k}$ *by*

$$H^+_{l,k} := \{F_{m,l,k} \mid m \leq -(2k+4l+n),\ m \equiv 2k+q \mod 4\ \}$$

Proposition 4. *Let* λ *be an admissible eigenvalue and* (l,k) *a* λ*-admissible pair. Then, as* $\mathfrak{sl}_2 \times O(n)$*-modules:*

(1) *If* $q \not\equiv n \mod 4$ *and* $q \not\equiv -n \mod 4$ *then* $H_{l,k}$ *is irreducible. Moreover, as an* $\mathfrak{sl}_2 \times O(n)$*-module* $\ker(\Omega'' - 2\lambda)_K$ *is decomposed as follows:*

$$\ker(\Omega'' - 2\lambda)_K = \bigoplus_{\substack{\lambda\text{-admissible}\\(l,k)}} H_{l,k}$$

(2) *If* $q \equiv n \mod 4$ *and* $q \not\equiv -n \mod 4$, *then* $H^+_{l,k}$ *is the unique irreducible* $\mathfrak{sl}_2 \times O(n)$*-submodule of* $H_{l,k}$.

(3) *If* $q \not\equiv n \mod 4$ *and* $q \equiv -n \mod 4$, *then* $H^-_{l,k}$ *is the unique irreducible* $\mathfrak{sl}_2 \times O(n)$*-submodule of* $H_{l,k}$.

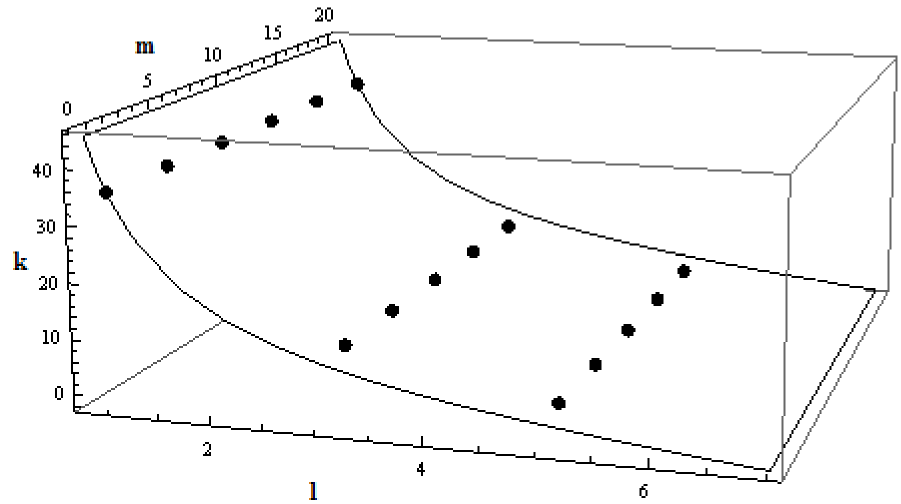

FIGURE 2. K-finite vectors of weight $\frac{m}{2}$ in $\ker(\Omega'' - 2\lambda) \subset I''(0, -3/2, s)$ for $0 \leq m \leq 20$, $n = 3$ and $\lambda = 75$.

(4) *If $q \equiv n \mod 4$ and $q \equiv -n \mod 4$, then $H^+_{l,k}$ and $H^-_{l,k}$ are the only irreducible $\mathfrak{sl}_2 \times O(n)$-submodules of $H_{l,k}$. A composition series for $H_{l,k}$ is*

$$0 \subset H^+_{l,k} \subset H^+_{l,k} \oplus H^-_{l,k} \subset H_{l,k}.$$

Proof. By Theorem 2, the representation is irreducible whenever $\pm(2k+4l+n) \neq m$ for any $m \equiv 2k+q \mod 4$, it has a highest or lowest weight submodule otherwise. □

Remark 2. *The direct sum in Proposition 4 is finite because, as a consequence of Proposition 1, the set of λ-admissible pairs (l, k) is finite for every admissible λ.*

When a lowest $H^+_{l,k}$, the representation is isomorphic to the $2k+4l+n$-th tensor product of the oscillator representation. Its dual occurs, when $H^-_{l,k}$ occurs.

Pictorially, $\ker(\Omega'' - 2\lambda)_K$ looks like depicted in Figure 2. There, we look at the case when $n = 3$ and $\lambda = 75$. On the (l, k)-plane the λ-level curve intersects the integral lattice, on three different points, $(5, 2)$, $(3, 9)$, and $(1, 36)$. These are the λ-admissible pairs for this particular case. At each of these pairs, we have an $\mathfrak{sl}_2 \times O(n)$ module represented by a line, where each dot corresponds to the complex span of $F_{m,l,k}$. Notice how on the m direction, the points are separated by jumps of 4 units. This corresponds to the action of $\eta^{\pm}$.

8. Heisenberg Action

In this section, we will calculate the action of the Heisenberg algebra on the K-finite vectors. As it turns out, elements of the Heisenberg algebra will take a K-finite vector $F_{m,l,k} \in \ker(\Omega'' - 2\lambda)$ and map it to a linear combination of K-finite vectors associated to possibly other eigenvalues. The fact that a pair $(l, k) \in \mathbb{Z}^{\geq 1} \times \mathbb{Z}^{\geq 0}$ determines a unique λ will be used to determine these eigenvalues explicitly. We

start stating a lemma that was proved in [4] and that will be used in the calculation of the actions below.

Lemma 3. *Let* $(k,n) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ *and define constants* $c_{k,n} = \frac{1}{2k+n-2}$ *for* $(k,n) \neq (0,2)$ *and* $c_{0,2} = 0$. *If* $1 \leq j \leq n$ *and* $h_k \in \mathcal{H}_k(\mathbb{R}^n)$, *then*

$$y_j h_k(y) - c_{k,n}\rho^2 \in \mathcal{H}_{k+1}(\mathbb{R}^n).$$

Moreover, if $h_k \neq 0$ *and* $(k,n) \neq (1,1)$, *then there exists a* $j \in \{1,2,...,n\}$ *for which* $y_j h_k(y) - c_{k,n}\rho^2(\partial_j h_k)(y) \neq 0$.

Let $\{e_j\}$ be the standard basis of $\mathbb{C}^n$ and define $E^{\mp} := (\pm i e_j, e_j, 0) \in \mathfrak{h}^{\mathbb{C}}_{2n+1}$. By Lemma 1, $E^{\mp}$ act on $I''(q, -\frac{n}{2}, s)$ by $e^{\pm i\theta}(\mp i\partial_j - 2sy_j)$.

Proposition 5. *For non-zero* $F_{m,l,k}$,

(8.1)
$$\begin{aligned} E_j^+.F_{m,l,k} = 2ilF_{m+2,l-1,k+1} - s\frac{(2l+2k+n-2)(m+2k+4l+n)}{2(k+2l+n/2-1)(k+2l+n/2)}F_{m+2,l,k+1} \\ + \frac{i}{2k+n-2}\left((2l+2k+n-2)F_{m+2,l,k-1}\right. \\ \left. + \frac{2isl(m+2k+4l+n)}{2(k+2l+n/2-1)(k+2l+n/2)}F_{m+2,l+1,k-1}\right) \end{aligned}$$

and

(8.2)
$$\begin{aligned} E_j^-.F_{m,l,k} = -s\frac{(2-2l-2k-n)(4l+2k+n-m)}{2(2l+k+n/2-1)}F_{m-2,l,k+1} - 2ilF_{m-2,l-1,k+1} \\ -\frac{i}{2k+n-2}\left((2l+2k+n-2)F_{m-2,l,k-1} - isl\frac{4l+2k+n-m}{2(2l+k+n/2-1)}F_{m-2,l+1,k-1}\right). \end{aligned}$$

Proof. Explicitly calculate

$$\begin{aligned} E_j^+.F_{m,l,k} = ip_{m+2}(\theta,\rho)\Big[(2l\rho^{2(l-1)}h_k y_j + \rho^{2l}\partial_j h_k)\,{}_1F_1(a,b,z) \\ + 4is\rho^{2l}y_j h_k \frac{a}{b}\ {}_1F_1(a+1,b+1,z)\Big]. \end{aligned}$$

Lemma 3 implies that there exists a possibly zero harmonic polynomial, $h_{k+1,j} \in \mathcal{H}_{k+1}(\mathbb{R}^n)$, such that $y_j h_k = h_{k+1,j} + c_{k,n}\rho^2\partial_j h_k$. Then, we can write

$$\begin{aligned} E_j^+.F_{m,l,k} = ip_{m-2}(\theta,\rho)\rho^{2(l-1)}\left(2l\,{}_1F_1(a,b,z) + 4is\rho^2\frac{a}{b}\,{}_1F_1(a+1,b+1,z)\right) \\ \cdot h_{k+1,j} + ic_{k,n}\rho^{2l}p_{m-2}(\theta,\rho)\Big((2l + c_{k,n}^{-1})\,{}_1F_1(a,b,z) \\ + 4is\rho^2\frac{a}{b}\,{}_1F_1(a+1,b+1,z)\Big)\partial_j h_k. \end{aligned}$$

Using (7.2) we obtain,

$$
\begin{aligned}
E_j^+.F_{m,l,k} &= i\rho^{2(l-1)}p_{m-2}(\theta,\rho)\,\Big(2l\,{}_1F_1(a,b-1,z)\\
&\qquad + \frac{az}{b}(2-\frac{l}{b-1})\,{}_1F_1(a+1,b+1,z)\Big)\,h_{k+1,j} + ic_{k,n}\rho^{2l}p_{m-2}(\theta,\rho)\\
&\quad\cdot\left((2l+c_{k,n}^{-1})\,{}_1F_1(a,b-1,z)+\frac{az}{b}(2-\frac{l+c_{k,n}^{-1}}{b-1})\,{}_1F_1(a+1,b+1,z)\right)\partial_j h_k,
\end{aligned}
$$

which gives

$$
\begin{aligned}
E_j^+.F_{m,l,k} &= 2ilF_{m+2,l-1,k+1} - s\frac{(2l+2k+n-2)(m+2k+2l+n)}{2(k+2l+n/2-1)(k+2l+n/2)}F_{m+2,l,k+1}\\
&\qquad + \frac{i}{2k+n-2}\Big((2l+2k+n-2)F_{m+2,l,k-1}\\
&\qquad\qquad + \frac{2isl(m+2k+2l+n)}{2(k+2l+n/2-1)(k+2l+n/2)}F_{m+2,l+1,k-1}\Big).
\end{aligned}
$$

The action of E^- is as follows:

$$
\begin{aligned}
E_j^-.F_{m,l,k} &= -i\rho^{2(l-1)}p_{m-2}(\theta,\rho)\left((2l-2z)y_jh_k+\rho^2\partial_jh_k\right)\,{}_1F_1(a,b,z)\\
&\qquad + 2\rho^{2(l-1)}p_{m-2}(\theta,\rho)z\frac{a}{b}y_jh_k\,{}_1F_1(a+1,b+1,z).
\end{aligned}
$$

Substitute $y_jh_k = h_{k+1,j}+c_{k,n}\rho^2\partial_jh_k$ and use Equation (7.1c) to obtain

$$
\begin{aligned}
E_j^-.F_{m,l,k} &= -i\rho^{2(l-1)}p_{m-2}(\theta,\rho)\,\big((2l+2-2b)\,{}_1F_1(a,b,z)\\
&\qquad +2(b-1)\,{}_1F_1(a-1,b-1,z)\big)\,h_{k+1,j} - ic_{k,n}\rho^{2l}p_{m-2}(\theta,\rho)\\
&\quad\cdot\left((2l+2-2b+c_{k,n}^{-1})\,{}_1F_1(a,b,z)+2(b-1)\,{}_1F_1(a-1,b-1,z)\right)\partial_jh_k.
\end{aligned}
$$

Using Equation (7.3) we obtain

$$
\begin{aligned}
E_j^-.F_{m,l,k} &= -i\rho^{2(l-1)}p_{m-2}(\theta,\rho)\,\Big((2l+2-2b)\frac{b-a}{b-1}z\,{}_1F_1(a,b+1,z)\\
&\quad +l\,{}_1F_1(a-1,b-1,z)\big)\,h_{k+1,j} - ic_{k,n}\rho^{2l}p_{m-2}(\theta,\rho)\,\Big((2l+2-2b+c_{k,n}^{-1})\\
&\qquad\cdot\frac{b-a}{b-1}z\,{}_1F_1(a,b+1,z)+(2l+c_{k,n}^{-1})\,{}_1F_1(a-1,b-1,z)\Big)\,\partial_jh_k.
\end{aligned}
$$

Writing out a and b explicitly, this is the desired result. □

Definition 3. *Let*

$$
H = \bigoplus_{\substack{k\geq 0\\ l\geq 1}} H_{l,k}.
$$

and $H_0 = \bigoplus_{k\geq 0} H_{0,k}$. *If* $n \equiv q \mod 4$, *define*

$$
H^+ = \bigoplus_{\substack{k\geq 0\\ l\geq 1}} H^+_{l,k}
$$

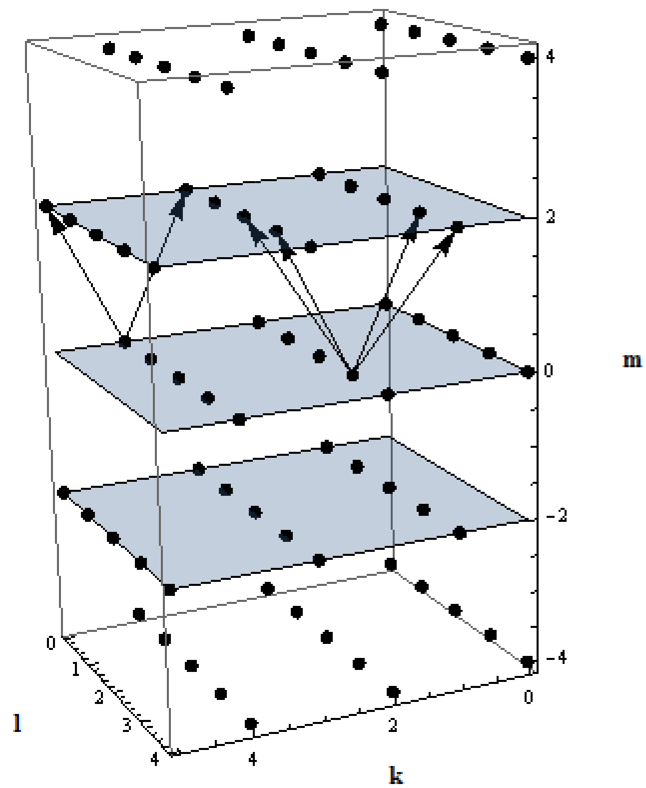

FIGURE 3. Action of E_j^+ on $H_0 \oplus H$

and $H_0^+ = \bigoplus_{k\geq 0} H_{0,k}^+$. *If* $n \equiv -q \mod 4$, *define*

$$H^- = \bigoplus_{\substack{k\geq 0\\ l\geq 1}} H_{l,k}^-.$$

and $H_0^- = \bigoplus_{k\geq 0} H_{0,k}^-$.

Remark 3. *In the previous definition, k runs over all non-negative integers. This is the case when $n \geq 3$. However, when $n = 2$, k runs over all the integers.*

In Figure 3 we show how the element E_j^+ in the Heisenberg algebra acts on the space of K-finite vectors.

Theorem 3. *As $\mathfrak{g}$ modules:*

(1) *If $q \not\equiv n \mod 4$ and $q \not\equiv -n \mod 4$ then H_0 is the unique irreducible submodule of H.*
(2) *If $q \equiv n \mod 4$ and $q \not\equiv -n \mod 4$, then a composition series for H is*

$$0 \subset H_0^+ \subset H_0 \subset H_0 \oplus H^+ \subset H$$

(3) *If* $q \not\equiv n \mod 4$ *and* $q \equiv -n \mod 4$*, then a composition series for* H *is*

$$0 \subset H_0^- \subset H_0 \subset H_0 \oplus H^- \subset H$$

(4) *If* $q \equiv n \mod 4$ *and* $q \equiv -n \mod 4$*, then a composition series for* H *is*

$$0 \subset H_0^- \subset H_0^+ \oplus H_0^- \subset H_0 \subset H_0 \oplus H^- \subset H_0 \oplus H^- \oplus H^+ \subset H$$

Proof. The statement in item (1) follows from noticing that by Proposition 5, when $l = 0$, the terms where the parameter l is changed are annihilated by the Heisenberg algebra. This, together with the fact that H_0 is irreducible under the action of $\mathfrak{g}$ (c.f. [4]) gives the result.

The proof of (2) and (3) is essentially the same. Therefore, we look at (2). The first two inclusions in the composition series are a consequence of Proposition 4 when $l = 0$. In order to show irreducibility of $H_0 \oplus H^+/H_0$ one has to notice that the actions of $E_j^\pm$ "respects" the highest weight structures. Let us elaborate further on this matter. Proposition 5 implies that

$$E_j^- .F_{(2k+4l+n),l,k} = -2ilF_{(2k+4l+n)-2,l-1,k+1} - i\frac{2l+2k+n-2}{2k+n-2}F_{(2k+4l+n)-2,l,k-1}$$

and this is a linear combination of highest weight vectors. In the same way, it can be seen from (8.1) that E_j^+ maps a highest weight vector to a linear combination of elements in the highest weight modules corresponding to the triples $(m+2, l-1, k+1)$, $(m+2, l, k-1)$, $(m+2, l, k+1)$, and $(m+2, l+1, k-1)$. The elements corresponding to the first two triples are highest weight vectors and the latter get mapped to one by the action of η^+. The rest of the composition series in (2) is clear.

□

Remark 4. *Suppose that* (l, k) *is a* λ*-admissible pair. Then, the action of* $E_j^\pm$ *sends* $F_{m,l,k}$ *to a linear combination of* $F_{m\pm2,l-1,k+1}$*,* $F_{m\pm2,l,k-1}$*,* $F_{m\pm2,l,k+1}$*, and* $F_{m\pm2,l+1,k-1}$*. However the pairs* $(l+1, k-1)$*,* $(l-1, k+1)$*,* $(l, k-1)$*, and* $(l, k+1)$ *are, in general, not admissible for* λ*, but they are admissible for different eigenvalues.*

Therefore, K*-finite vectors in* $\ker(\Omega'' - 2\lambda)$ *get sent, by* $E_j^\pm$*, to a linear combination of* K*-finite vectors in* $\ker(\Omega'' - 2(\lambda \pm (2k+2l+n-2)))$ *and in* $\ker(\Omega'' - 2(\lambda \pm l))$*.*

References

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Earl A. Coddington. *An introduction to ordinary differential equations*. Prentice-Hall Mathematics Series. Prentice-Hall Inc., Englewood Cliffs, N.J., 1961.

[3] Jose Franco. Global $\widetilde{SL(2, \mathbb{R})}$ representations of the schrödinger equation with time-dependent potentials.

[4] Mark R. Sepanski and Ronald J. Stanke. Global Lie symmetries of the heat and Schrödinger equation. *J. Lie Theory*, 20(3):543–580, 2010.

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